Semester Exam - Representation Theory of groups
Maximum Time: 3hrs - Maximum Marks: 50
Section I: Each question is worth 1 mark - Maximum Marks: 5

1. Is $GL(n, \mathbb{R})$ connected? Justify your answer.

2. Prove that a continuous homomorphism of a connected topological group into $\mathbb{R}$ is onto.

3. Let G be a locally compact group such that $G/\overline{[G,G]}$ is compact. Then prove that G is unimodular.

4. Let π be a non-trivial irreducible representation of a finite group G on a vector space V over $\mathbb{C}$. Then show that $\sum_{g \in G} \pi(g) = 0$.

5. Show that any one-dimensional unitary representation of the Heisenberg group $\mathbb{H}_1$ is trivial on the center of $\mathbb{H}_1$.

Section II: Answer any 4
Each question is worth 5 marks - Maximum Marks: 20

1. Let G be a topological group and G_0 be the connected component of e in G. Then show that G_0 is a closed normal subgroup of G such that G/G_0 is totally disconnected and any open subgroup of G contains G_0.

2. Let ρ be the permutation representation associated to the action of a finite group G on a finite set X. Then show that $\chi_\rho(g)$ is the number of elements fixed by g.

3. Prove that a unitary representation π of a locally compact group G is irreducible if and only if $\mathcal{C}(\pi)$ contains only scalar multiples of identity and deduce that π is one-dimensional if G is abelian and π is irreducible.

4. Show that any finite-dimensional unitary representation π of the affine group $\mathbb{R}^+ \ltimes \mathbb{R}$ is of the form $\pi(a, b) = a^{ir}$ for some $r \in \mathbb{R}$.

5. For $g \in SU(n)$, let $f(g) = \mathrm{tr}(g)$. Show that $f \in L_2$ of norm one and $f * f = \frac{f}{n}$. Show that the center of $SU(n)$ has n-elements if $n \geq 2$.

6. Let V be the Hilbert space of all homogeneous polynomials of degree $m \geq 2$ in two complex variables. Let π be the canonical representation of $SU(2)$ on V. Show that π is unitary and irreducible.

Section III: Answer any two
Each question is worth $12\frac{1}{2}$ marks - Maximum Marks: 25

1. (a) Give a counter-example to show that continuous homomorphism of locally compact groups need not be open. Justify your answer.

   (b) Let G be a finite group and R be the left-regular representation of G on $V = \mathbb{C}[G]$. Let α be any $\mathbb{C}$-valued function on G and $\phi: V \to V$ be $\phi = \sum_{g\in G} \alpha(g)R(g)$. Then ϕ is a G-map if and only if α is a class function.

2. Let G be a locally compact group with a right-invariant Haar measure m. Then prove the following:

   (a) for any Borel set E, $m(E) = 0$ if and only if $m(E^{-1}) = 0$;

   (b) G is discrete if and only if $m(\{e\}) > 0$;

   (c) G is compact if and only if m is finite.

3. (a) Prove that right and left regular representations of a locally compact group are unitarily equivalent.

   (b) Prove Gelfand-Raikov Theorem for compact groups.